Grundy number on P4-classes

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Abstract

In this article, we define a new class of graphs, the fat-extended P_4-laden graphs, and we show a polynomial time algorithm to determine the Grundy number of the graphs in this class. This result implies that the Grundy number can be found in polynomial time for any graph of the following classes: P_4-reducible, extended P_4-reducible, P_4-sparse, extended P_4-sparse, P_4-extendible, P_4-lite, P_4-tidy, P_4-laden and extended P_4-laden, which are all strictly contained in the fat-extended P_4-laden class.

Keywords: Graph Theory, Grundy number, P_4-classes, Modular decomposition

1 Introduction

Given a graph $G = (V, E)$, a k-coloring of G is an assignment of k colors to the vertices of G in such a way that adjacent vertices have distinct colors. The chromatic number of G, $\chi(G)$, is the minimum integer k such that G admits a k-coloring. The problem of determining $\chi(G)$ is NP-hard [6]. Therefore, the evaluation of the performance of fast vertex coloring algorithms is a relevant problem. The greedy coloring algorithm is a linear vertex coloring algorithm. Given an order $\theta = v_1, \ldots, v_n$ over V, the greedy algorithm to color the vertices of G assigns to v_i the minimum positive integer that was not already assigned to its neighbors in the set $\{v_1, \ldots, v_{i-1}\}$. A coloring obtained by an execution of this algorithm is usually called as a greedy coloring.

The maximum number of colors of a greedy coloring of a graph G, over all the orders θ of $V(G)$, is the *Grundy number* of G and it is denoted by $\Gamma(G)$.

Determining the Grundy number is NP-complete even for complements of bipartite graphs [10]. In fact, given a graph G and an integer r it is a $coNP$-complete problem to decide if $\Gamma(G) \leq \chi(G)+r$ [10] or if $\Gamma(G) \leq r\times\chi(G)$ or if $\Gamma(G) \leq c\times\omega(G)$ [2], where c is a constant and $\omega(G)$ is the size of a maximum clique of G. However, there are polynomial time algorithms

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to calculate the Grundy number of the following classes of graphs: cographs [4], trees [5] and k-partial trees [9].

2 Fat extended P_4-laden graphs

We start this section by introducing some definitions. Let $G = (V, E)$ be a graph and S a subset of $V(G)$. We denote by $G[S]$ the subgraph of G induced by S. We say that M is a *module* of a graph G if, for every vertex w in $V \backslash M$ and every pair of vertices x and y in M, either w is adjacent to both x and y or w is not adjacent to both x and y. The sets V and $\{x\}$, for every $x \in V$, are *trivial modules*, the last one being called as a *singleton* module. A graph is *prime* if all its modules are trivial. We say that M is a *strong module* of G if, for every module M' of G, either $M' \cap M = \emptyset$ or $M \subset M'$ or $M' \subset M$. The modular decomposition is a form of decomposition of a graph G that associates with G a unique *modular decomposition tree* $T(G)$. The leaves of $T(G)$ are the vertices of G and a set of leaves of $T(G)$ having the same least common ancestor in $T(G)$ is a strong module of G. Let r be an internal node of $T(G)$, $M(r)$ be the set of leaves of the subtree of $T(G)$ rooted on r, and $V(r) = \{r_1, \ldots, r_k\}$ be the set of children of r in $T(G)$. If $G[M(r)]$ is disconnected, then r is called a *parallel* node and $G[M(r_1)], \ldots, G[M(r_k)]$ are its components. If $\bar{G}[M(r)]$ is disconnected then r is called a *series* node and $\bar{G}[M(r_1)], \ldots, \bar{G}[M(r_k)]$ are the components of $\bar{G}[M(r)]$. Finally, if both graphs $G[M(r)]$ and $\bar{G}[M(r)]$ are connected, then r is called a *neighborhood* node and $\{M(r_1), \ldots, M(r_k)\}$ is the unique set of maximal strong proper submodules of $M(r)$. The *quotient* graph of r, denoted by $G(r)$, is $G[\{v_1, \ldots, v_k\}]$, where $v_i \in M(r_i)$, for $1 \leq i \leq k$. We say that r is a *fat* node, if $M(r)$ is not a singleton module.

A graph is a *spider* if its vertex set can be partitioned into three sets S, K and R in such a way that S is a stable set, K is a clique, all the vertices of R are adjacent to all the vertices of K and to none of the vertices of S and there is a bijection $f : S \to K$ such that, for all $s \in S$, either $N(s) = f(s)$ (and it is a *thin spider*) or $N(s) = K - f(s)$ (and it is a *fat spider*).

A graph G is split if and only if it is $\{C_5, C_4, \bar{C}_4\}$-free, which is equivalent to say that $V(G)$ can be partitioned in two sets S and K such that S is a stable set and K is a clique. A *pseudo-split* graph is defined as a $\{C_4, \bar{C}_4\}$-free graph. Moreover, given a split graph $G = (S \cup K, E)$, its vertex set can be partitioned into three disjoint sets $S(G)$, $K(G)$ and $R(G)$ such that $S(G)$ is composed by all the vertices of S which are not adjacent to at least one vertex in K, $K(G)$ is the neighborhood of the vertices in $S(G)$ and $R(G) = V(G) \backslash \{S(G) \cup K(G)\}$.

Giakoumakis [3] defined a graph G as *extended P_4-laden graphs* if, for all $H \subseteq G$ such that $|V(H)| \leq 6$, then the following statement is true: if H contains more than two induced P_4's, then H is a pseudo-split graph. An extended P_4-laden graph can be completely characterized by its modular decomposition tree, as follows:

Theorem 2.1 **([3])** *Let $G = (V, E)$ be a graph, $T(G)$ be its modular decomposition tree and r be any neighborhood node of $T(G)$, with children $r_1, \ldots, r_k$. Then G is extended P_4-laden if and only if $G(r)$ is isomorphic to:*

(i) *a P_5 or a $\bar{P}_5$ or a C_5, and each $M(r_i)$ is a singleton module; or*

(ii) *a spider $H = (S \cup K \cup R, E)$ and each $M(r_i)$ is a singleton module, except the one corresponding to R and eventually another one which may have exactly two vertices; or*

(iii) *a split graph H, whose modules corresponding to the vertices of $S(H)$ are independent sets and the ones corresponding to the vertices of $K(H)$ are cliques.*

We say that a graph is **fat-extended P_4-laden** if its modular decomposition satisfies the Theorem 2.1, except in the first case, where $G(r)$ is isomorphic to a P_5 or a $\bar{P}_5$ or a C_5, but the maximal strong modules $M(r_i)$, $1 \leq i \leq 5$, of $M(r)$ are not necessarily singleton modules.

3 Grundy number on fat extended P_4-laden graphs

From now, let $G = (V, E)$ be a fat-extended P_4-laden graph and $T(G)$ its modular decomposition tree. Since $T(G)$ can be found in linear time [8], we propose an algorithm to calculate $\Gamma(G)$ that uses a bottom-up strategy. We know that the Grundy number of the leaves of $T(G)$ is equal to one and we show in this section how to determine the Grundy number of $G[M(v)]$, for every inner node v of $T(G)$, based on the Grundy number of its children.

First, observe that for every series node (resp. parallel node) v of $T(G)$, the Grundy number of $G[M(v)]$ is equal to the sum of the Grundy number of its children (resp. the maximum Grundy number of its children) [4]. Thus, we only need to prove that the Grundy number of $G[M(v)]$ can be found in polynomial time when v is a neighborhood node of $T(G)$.

The following result is a simple generalization of a result due to Asté et al. [2] for the Grundy number of lexicographic product of graphs:

Proposition 3.1 *Let G, $H_1, \ldots, H_n$ be disjoint graphs such that $n = |V(G)|$ and let $V(G) = \{v_1, \ldots, v_n\}$. Let G' be the graph obtained by replacing $v_i \in V(G)$ by H_i, in such a way that there exist all the edges between the vertices of H_i and H_j, $i \neq j$, if and only if $v_iv_j \in E(G)$. Then for every greedy coloring of G' at most $\Gamma(H_i)$ colors contain vertices of the induced subgraph $G'[H_i] \subseteq G'$, for all $i \in \{1, \ldots, n\}$.*

Before presenting the next lemma, observe that a greedy k-coloring of G can be viewed as a partition $\mathcal{S} = \{S_1, \ldots, S_k\}$ of $V(G)$ in such a way that every vertex in S_j has at least one neighbor in the color class S_i, for all $j > i$, $i, j \in \{1, \ldots, k\}$.

Lemma 3.2 *Let v be a neighborhood node of $T(G)$ such that $G(v)$ is isomorphic to a P_5 or a C_5 or a $\bar{C}_5$, $v_1, \ldots, v_5$ be the children of v and Γ_i be the Grundy number of $G[M(v_i)]$, $1 \leq i \leq 5$. Then $\Gamma(G[M(v)])$ can be found in constant time.*

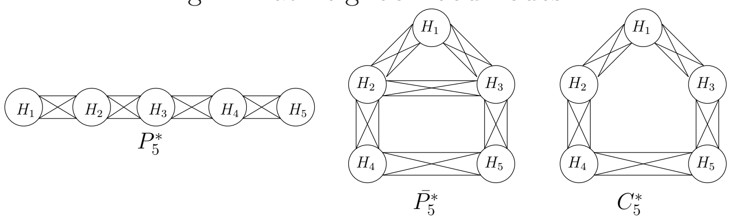
Proof (Sketch) Without loss of generality, suppose that $v_1, \ldots, v_5$ label the children of v as depicted in Figure 1 and $\Gamma_i = \Gamma(G[M(v_i)])$. In order to simplify the notation, denote by θ_i an ordering over $M(v_i)$ that induces a greedy coloring with Γ_i colors, $1 \leq i \leq 5$.

We calculate $\Gamma(G[M(v)])$ by verifying all the possible configurations for a greedy $\Gamma(G[M(v)])$-coloring and by returning the greater value found between all the cases. Suppose that $G(v)$ is isomorphic to a P_5. Let $\mathcal{S} = \{S_1, \ldots, S_k\}$ be a greedy $\Gamma(G[M(v)])$-coloring of $G[M(v)]$.

We claim that if there exists a vertex $u \in V(H_1)$ colored by S_k, then $\Gamma(G[M(v)]) = \Gamma_1 + \Gamma_2$. This fact holds because combining the observation that u has at least one vertex colored by S_i, for all $i \in \{1, \ldots, k-1\}$, with the Proposition 3.1, we conclude that $\Gamma(G[M(v)]) \leq \Gamma_1 + \Gamma_2$. On the other hand, if we consider any ordering θ over $G[M(v)]$ that has starts with θ_1 and θ_2, we see that the first-fit algorithm over this order will produce a greedy coloring with at least $\Gamma_1 + \Gamma_2$ colors. Using the symmetry, we can also prove that if $u \in V(H_5)$, then $\Gamma(G[M(v)]) = \Gamma_4 + \Gamma_5$.

All the other cases use similar arguments, that is, by finding an upper bound based on the position of a vertex colored S_k and a lower bound based in an ordering over $M(v)$. The cases where $G(v)$ is isomorphic to C_5 or $\bar{P}_5$ are also proved by using similar arguments. $\square$

Fig. 1. Fat neighborhood nodes.



Lemma 3.3 *Let v be a neighborhood node of $T(G)$ such that $G(v)$ is isomorphic to a spider $H = (S \cup K \cup R, E)$, f_r be its child corresponding to R, f_2 be its child corresponding to the module which has eventually two vertices and $\Gamma(R)$ be the Grundy number of $G[M(f_r)]$. Then $\Gamma(G[M(v)])$ can be found in $\mathcal{O}(V(G[M(v)]))$.*

Proof (Sketch) If $M(f_2)$ is singleton module, then $G[M(v)]$ is a spider. In this case, we cannot have two colors S_i and S_j, $j > i$, such that both contain only vertices of S. For otherwise, since S is a stable set, the vertices colored S_j would not any neighbor colored S_i, a contradiction. Thus, $\Gamma(G[M(v)]) \leq 1+|K|+\Gamma(R)$. If $R = \emptyset$, then an ordering over $M(v)$ such that all the vertices of S come before the vertices of K induces a greedy coloring with $\Gamma(G[M(v)]) = 1+|K|$ colors. If $R \neq \emptyset$, we will prove that $\Gamma(G[M(v)]) \leq |K|+\Gamma(R)$. Observe first that there is at least one color S_i occurring in R. Consequently, S_i does not occur in K. Thus, there is no order over $M(v)$ whose greedy coloring returns a color S_j containing only vertices of S, because a vertex of S colored S_j would not be adjacent to a vertex colored S_i. On the other hand, if θ_R is an ordering that induces a greedy $\Gamma(R)$-coloring of R, then any ordering over $M(v)$ starting by θ_R induces a greedy coloring with at least $|K|+\Gamma(R)$ colors. The case where $M(f_2)$ is not a singleton module is proved using similar arguments. $\square$

If $G(v)$ is a split graph H and the factors corresponding to vertices of $S(H)$ are independent sets and the ones corresponding to vertices of $K(H)$ are cliques, then we can use the same arguments of Lemma 3.3 observing that S, K and R correspond to $S(H)$, $K(H)$ and $R(H)$, respectively.

Theorem 3.4 *If $G = (V, E)$ is a fat-extended P_4-laden graph and $|V| = n$, then $\Gamma(G)$ can be found in $\mathcal{O}(n^3)$.*

Proof. The algorithm calculates $\Gamma(G)$ by traversing the modular decomposition tree of G in a postorder way and determining the Grundy of each inner node of $T(G)$ based on the Grundy number of the leaves. The modular decomposition tree can be found in linear time, the postorder traversal can be done in $\mathcal{O}(n^2)$-time and the Grundy number of each inner node can be found in linear time on the number of vertices of the corresponding module, because of Lemmas 3.2 and 3.3 and the results of Gyárfás and J. Lehel [4] for cographs. $\square$

Corollary 3.5 *Let G be a graph that belongs to one of the following classes: P_4-reducible, extended P_4-reducible, P_4-sparse, extended P_4-sparse, P_4-lite, P_4-extendible, P_4-tidy, P_4-laden and extended P_4-laden. Then, $\Gamma(G)$ can be found in polynomial time.*

Proof. According to definition of these classes [7], they are all strictly contained in the fat-extended P_4-laden graphs and so the corollary follows. $\square$

The complete proofs of the results in this paper can be found in [1].

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